

Exercise Sheet 1

Exercise 1. (for credit, due on 21 September)

(5 points) Prove that the Riemann surface \mathbb{C}/\mathbb{Z} is biholomorphic to $\mathbb{C}\setminus\{0\}$.

Exercise 2. Prove that the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ are biholomorphic.

Exercise 3. (challenging)

Let $\mathcal{S} = \{f : \mathbb{D} \rightarrow \mathbb{C} \text{ holomorphic, injective, } f(0) = 0, f'(0) = 1\}$.

(a) Show that the map

$$k : \mathbb{D} \rightarrow \mathbb{C} \setminus (-\infty, -1/4], \quad z \mapsto \frac{z}{(1-z)^2}$$

is a biholomorphism and $k \in \mathcal{S}$.

(b) If $f(z) = z + \sum_{n \geq 2} a_n z^n \in \mathcal{S}$, prove that $|a_2| \leq 2$ and that equality holds for the function k .

(c) Prove that every $f \in \mathcal{S}$ satisfies $B_{1/4}(0) \subset f(\mathbb{D})$ and that the constant $1/4$ cannot be increased.

Remark. The Bieberbach conjecture, proved in 1985 by de Branges, states that if $f(z) = z + \sum_{n \geq 2} a_n z^n \in \mathcal{S}$, then $|a_n| \leq n$ for all $n \geq 2$.

Exercise 4. Let X and Y be Riemann surfaces. Assume that X is connected and compact.

(a) Let $f : X \rightarrow Y$ be a holomorphic and non-constant map. Show that f surjects onto a compact connected component of Y .

(b) Deduce from (a) that every holomorphic map $f : X \rightarrow \mathbb{C}$ is constant.

(c) Deduce from (a) the fundamental theorem of algebra: a non-constant complex polynomial has at least one root.

Exercise 5. Let $\Omega \subseteq \mathbb{C}$ be an open subset of the complex plane. A meromorphic function on Ω can be defined as a function f which is holomorphic outside a discrete subset $S = \{p_i, i \in I\} \subseteq \Omega$ and admits the following Laurent expansion in a neighborhood of p_i ($i \in I$):

$$(1) \quad f(z) = \frac{a_{-N}}{(z-p_i)^N} + \dots + \frac{a_{-1}}{z-p_i} + a_0 + a_1(z-p_i) + \dots$$

(1) Show that a meromorphic function on Ω is the same as a holomorphic function $\Omega \rightarrow \mathbb{C}\mathbb{P}^1$ (which is not identically ∞ on any connected component).

(2) A function of the form $R(z) = P(z)/Q(z)$, where P and Q are polynomials, is called a rational function. Show that a rational function defines a meromorphic function on $\mathbb{C}\mathbb{P}^1$.

(3) Prove that every meromorphic function on $\mathbb{C}\mathbb{P}^1$ is rational.